

Ergodicity criterion for one-sided, one-dimensional IPS with a long-lived state*

Maciej Gluchowski[†]  Georg Menz[‡] 

Abstract

We study one-dimensional, one-sided, nearest-neighbor Interacting Particle Systems (IPS) with positive rates and identify a criterion for ergodicity based on the presence of a *long-lived state* a site can occupy. The criterion is that the IPS admits a state with a ratio of rates of exiting to entering below $\sqrt{2}$. The main idea is that under the canonical coupling, it is possible to identify large blocks in spacetime where two coupled trajectories must agree to share the special state, which inhibits the spread of disagreement. The result covers much of the parameter region left open by earlier work on the Positive Rates Conjecture (PRC) for simple IPS, narrowing the unresolved IPS to noisy versions of the East model.

Keywords: interacting particle system; stochastic process; ergodicity; invariant measure; positive rates conjecture; East model.

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1 Introduction

Interacting particle systems (IPS) provide a class of toy models that are simple enough to allow for rigorous mathematical analysis and yet are subtle enough to show a plethora of interesting physical phenomena, e.g. phase transitions. The broader theme of this article is the question of whether a given IPS is ergodic, which means, in layman’s terms, that it will forget its initial state over time.

To be more precise, an IPS (in this exposition exclusively with single-site updates) is a Markov process on the space of configurations $\mathcal{A}^\Lambda$ for some finite alphabet of possible states $\mathcal{A}$ (usually $\{0, 1\}$) and lattice Λ (usually $\mathbb{Z}^d$) where updates occur independently at each site, triggered by exponential clocks with (arbitrary) rate one. Once a clock rings at some $j \in \Lambda$, the process has an opportunity to change the state at that site depending on the states of the site’s neighbors $\mathcal{N}_j$. This neighborhood is usually $\mathcal{N}_j = j + \mathcal{N}$, with $\mathcal{N}$ a finite subset of $\mathbb{Z}^d$. These transitions are governed by transition rates $c_j(a, \zeta)$ or, equivalently, by transition kernel $P_j(a | \zeta)$. The latter is the convention we use and is to be interpreted as the probability for site j to pick its new state as $a \in \mathcal{A}$

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[†]University of California, Los Angeles, United States of America. E-mail: mgluchowski@math.ucla.edu

[‡]University of California, Los Angeles, United States of America.

E-mail: gmenz@math.ucla.edu <https://www.math.ucla.edu/people/ladder/gmenz>

given the trajectory is in configuration $\zeta \in \mathcal{A}^\Lambda$ immediately before an update at j . The transition kernel has finite range, meaning each function $P_j(a \mid \zeta)$ must depend on ζ only through $\zeta \upharpoonright_{\mathcal{N}_j}$. We identify the IPS with its transition kernel and write $\text{IPS}(P)$ to refer to the IPS with kernel $\{P_j\}_{j \in \Lambda}$. For a reader new to IPS, we point to an introductory paper by Durrett [4], and the textbook on IPS by Liggett [9]. For precise definitions and the construction of the IPS we study, we refer to the companion article [6].

The main idea behind the result is to combine the canonical coupling with the presence of a long-lived state to force local coalescence of the coupled processes. Recall that under the canonical coupling (defined in Section 4 and in more detail in Chapter 3, Section 1 of [9]), all initial configurations are driven by the same Poisson update times and auxiliary randomness. In such a coupling, discrepancies between two coupled copies can only propagate through the spacetime cone of dependence. Since ergodicity of a one-dimensional finite-range IPS is a local property, it is enough to show that for every fixed finite region $S \subset \mathbb{Z}$ any two coupled trajectories eventually coincide on S . Indeed, once local coalescence holds, the distribution of the coupled trajectories on S matches, meaning it's independent from the initial configuration. This in turn implies ergodicity.

Long-lived states provide a mechanism to force this coalescence. Suppose that for some state $a \in \mathcal{A}$ the transition kernel is such that a is born with uniformly positive probability and, once present, is unlikely to die. In the canonical coupling this yields random spacetime agreement blocks: whenever a site is forced into the state a (simultaneously for all coupled copies) and remains there for a while, all coupled trajectories are synchronized on that entire spacetime region. Such blocks act as barriers for the spread of disagreements: while a block is present at a site, discrepancies cannot cross it, and in one dimension this effectively separates the evolution on its left from the evolution on its right. Under positive rates these barriers are created with positive frequency, so the region where disagreements may persist becomes trapped between two moving boundaries. Controlling these boundaries reduces to a one-dimensional random-walk problem: if the maximal rate at which barriers disappear is sufficiently small compared to the minimal rate at which they appear, then the size of the “disagreement interval” has negative drift and collapses. A direct computation shows that the relevant ratio threshold is $\sqrt{2}$, leading to the criterion stated in Theorem 3.1.

We apply the result to continue our previous work in Chapter 7 of [6], where we focus on the problem of proving that all *simple* IPS with positive rates are ergodic. An IPS is simple if it is nearest-neighbor, one-sided, one-dimensional, with two states. The problem is reduced to a small region in the four-dimensional parameter space, where no known proof methods apply. The main result of this work is an ergodicity criterion in the form of a simple inequality, which covers most of the remaining region. The only simple IPS with positive rates for which the determination of ergodicity remains an open problem are IPS acting like a noisy version of the East model.

Random-walk-based coupling arguments are standard for determining properties of IPS or Markov processes in general. Examples include the celebrated proof of PRC for attractive, nearest-neighbor IPS due to Gray [7] or Durrett's study of drift of the supercritical contact process [3].

2 Positive rates conjecture for simple IPS

This article is a continuation of our work in Chapter 7 of [6], where we investigate the simplest non-trivial case of the celebrated Positive Rates Conjecture (PRC). To elaborate, we need to recall the definitions of *uniform*, *exponential ergodicity* and *positive rates*.

Definition 2.1 (Ergodicity). *An $\text{IPS}(P)$ is uniformly, exponentially ergodic if it has a*

unique invariant probability measure μ and there exist $C, \gamma > 0$ such that

$$\sup_{\eta \in \mathcal{A}^\Lambda} \|\delta_\eta P_t - \mu\|_{TV} \leq Ce^{-\gamma t}.$$

Definition 2.2 (Positive rates). *An IPS(P) has positive rates if*

$$\inf \{P_j(a \mid \zeta) : j \in \Lambda, a \in \mathcal{A}, \zeta \in \mathcal{A}^\Lambda, \zeta(j) \neq a\} > 0.$$

Intuitively the positive rates property means that the probability of a site to transition between any two states is uniformly positive. The probability is allowed to be zero for the non-transition because it is indistinguishable from there being no update at all. This introduces noise to the system, which in the one-dimensional, homogeneous (P is translation invariant) setting is conjectured to be ergodic, i.e. prevents the IPS from remembering its original configuration. This is the subject of the PRC.

Conjecture 2.3 (Positive Rates Conjecture). *Every IPS on a one-dimensional lattice with homogeneous interactions of bounded range and positive rates is ergodic.*

While there exists a counterexample to PRC published by Gács in [5] which has an alphabet $\mathcal{A}$ of size 2^{293} , we believe that the result holds for sufficiently basic IPS. The simplest class of IPS, for which the question of ergodicity is non-trivial, is termed so.

Definition 2.4 (Simple IPS). *An IPS is simple if it is two-state, one-dimensional, homogeneous with one-sided nearest-neighbor interactions, meaning:*

$$\mathcal{A} = \{0, 1\}, \Lambda = \mathbb{Z}, \mathcal{N}_j = \{j, j+1\}, P_j(\cdot \mid \zeta) = P_0(\cdot \mid T_j(\zeta)).$$

Here $\{T_j\}_{j \in \mathbb{Z}}$ are the translation operators, defined via $T_j(\zeta)(i) := \zeta(i+j)$. For convenience we later write $P(\cdot \mid \zeta(0)\zeta(1))$ instead of $P_0(\cdot \mid \zeta)$.

Conjecture 2.5 (PRC for simple IPS). *Every simple IPS with positive rates is ergodic.*

For the classification of simple IPS and examination of simulations that support the conjecture, we refer the reader to Chapter 3 of [6]. Here, we briefly examine the current state of the conjecture systematically. A simple IPS is determined by four parameters

$$p_{1|11} := P(1 \mid 11), \quad p_{1|10} := P(1 \mid 10), \quad p_{1|01} := P(1 \mid 01), \quad p_{1|00} := P(1 \mid 00).$$

The positive rates property is equivalent to

$$p_{1|11} < 1, \quad p_{1|10} < 1, \quad p_{1|01} > 0, \quad p_{1|00} > 0.$$

A consequence of the Time-Scaling Lemma (Lemma 4.1 in [6]) is that for $\lambda \in (0, 1]$ the ergodicity of a simple IPS(P) is equivalent to ergodicity of the IPS with parameters

$$\begin{aligned} P^\lambda(1 \mid 11) &= 1 - \lambda + \lambda p_{1|11}, & P^\lambda(1 \mid 10) &= 1 - \lambda + \lambda p_{1|10}, \\ P^\lambda(1 \mid 01) &= \lambda p_{1|01}, & P^\lambda(1 \mid 00) &= \lambda p_{1|00}. \end{aligned}$$

For any IPS(P) consider the line in the parameter space $[0, 1]^4$ passing through P and I . All IPS on this line define the same process up to time-scaling, with IPS closer to I being slowed down. For every IPS there exists its least slowed down version, whose parameters lie on one of the three-dimensional faces of $[0, 1]^4$, located on the opposite side from I . The question of simple PRC hence reduces to the verification of the conjecture for parameters on those faces, i.e., that lie in the union of four cubes

$$\{p_{1|11} = 0\} \cup \{p_{1|00} = 1\} \cup \{p_{1|10} = 0\} \cup \{p_{1|01} = 1\}.$$

Further simplification is possible since switching the roles of states 0 and 1 does not affect the dynamics of the IPS. Modulo this transformation $\{p_{1|11} = 0\}$ is equivalent

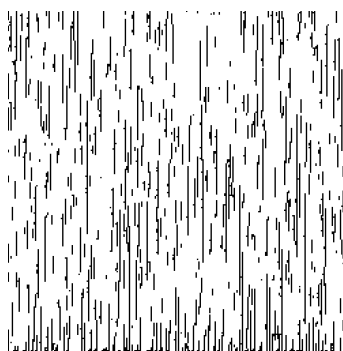


Figure 1: Simulated trajectory of the IPS with parameters $p_{1|11} = 0$, $p_{1|10} = 0.9$, $p_{1|01} = 0.02$, $p_{1|00} = 0.02$. Black represents state 1, white state 0. The arrow of time points up. The initial configuration is random, sampled uniformly. We observe that while 1s annihilate each other, the state 0 is almost stationary, building *walls* blocking the flow of information. Because of this typical shape of the trajectories, those IPS were called *wall IPS* in the companion article [6].

to $\{p_{1|00} = 1\}$ and $\{p_{1|10} = 0\}$ is equivalent to $\{p_{1|01} = 1\}$. Thus, the simple PRC is reduced to the cases

$$\{p_{1|11} = 0\} \cup \{p_{1|10} = 0\}.$$

In Chapter 7 of [6] we show that PRC holds for IPS with parameters in the cube $\{p_{1|10} = 0\}$ and for the region of the $\{p_{1|11} = 0\}$ cube that intersects with

$$\left\{p_{1|10} < \frac{1}{2}, p_{1|01} > 0, p_{1|00} > 0\right\} \cup \{p_{1|10} < p_{1|01} + p_{1|00}\} \cup \{p_{1|01} < p_{1|00}\}. \quad (2.1)$$

For an illustration of the covered and remaining regions, we refer to Figure 2.

3 Main result and discussion

In the main result of this work, we show that if there is a state $a \in \mathcal{A}$ that is born sufficiently more frequently than it dies, then the IPS is ergodic.

Theorem 3.1 (Main result). *For a homogeneous, one-dimensional, one-sided, nearest-neighbor IPS(P) with arbitrary alphabet $\mathcal{A}$ and a fixed $a \in \mathcal{A}$ define coefficients*

$$\begin{aligned} \beta &:= \inf_{\zeta} P_0(a \mid \zeta), \\ \delta &:= \sup_{\zeta: \zeta(0)=a} 1 - P_0(a \mid \zeta). \end{aligned}$$

These coefficients stand for the birth and death rates of the long lived state a . If $\delta < \sqrt{2}\beta$ then the IPS is uniformly exponentially ergodic.

Heuristically, the statement of the last theorem is plausible as the state a dominates the trajectories of the IPS, which inhibits the spread of information between distant sites. For an illustration, we refer to Figure 1.

Due to Theorem 3.1, we can update the region of simple IPS where PRC holds (Fig. 2). The region for which ergodicity has not yet been proven lies next to the boundary

$$\{p_{1|11} = 0, p_{1|10} = 1, p_{1|01} > 0, p_{1|00} = 0\},$$

which corresponds, after an application of the time-scaling lemma, to the set

$$\{p_{1|11} = p_{1|01} > 0, p_{1|10} = 1, p_{1|00} = 0\}.$$

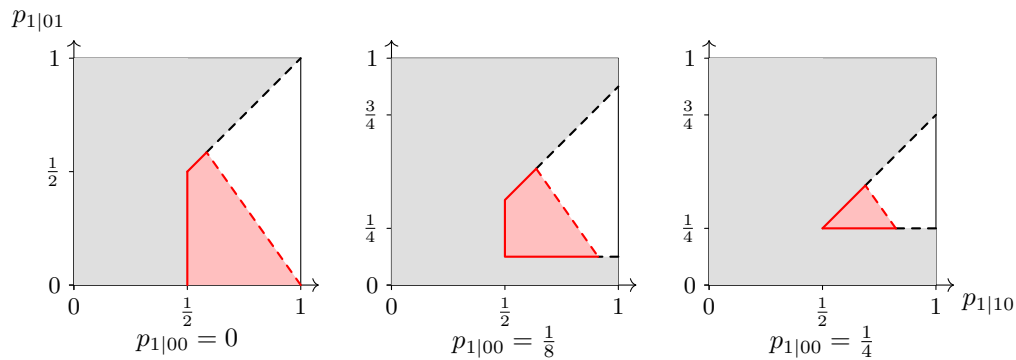


Figure 2: Sections of the cube $\{p_{1|11} = 0\}$ with $p_{1|00} \in \{0, \frac{1}{8}, \frac{1}{4}\}$. The parameter $p_{1|10}$ varies on the X axis and the parameter $p_{1|01}$ on the Y axis. The grey region represents the IPS for which ergodicity has been proven previously, i.e. the set 2.1. The three components are best seen in the middle graph. The red marks the new parameters corresponding to the IPS to which Theorem 3.1 applies, i.e. $\delta < \sqrt{2}\beta$. Dotted lines imply the boundary is not included. For $p_{1|00} > \frac{1}{2}$ the entire square is in the grey region. The line $\{1\} \times [0, 1]$ in the case $p_{1|00} = 0$ corresponds to the East model, after an application of the time-scaling lemma.

The last set corresponds to the one-dimensional East model, which is the classic example of a Kinetically Constrained Spin Model. The one-dimensional East model, with parameter $q \in (0, 1)$, is not ergodic, having multiple invariant measures, e.g. δ_0 and $\pi_q = \otimes_{\mathbb{Z}} \text{Bern}(q)$. This is not a problem for the PRC for simple IPS, as the East model does not have positive rates. However, the East model is almost ergodic: The δ_0 measure is unstable, in the sense that the process starting from any configuration with infinite ones to the left converges to π_q (Theorem 3.1 in [2]). The IPS also has a positive spectral gap w.r.t. π_q . For proof see Theorem 4.2 in [1] (note switched roles of 0 and 1). Due to these properties, the addition of any noise immediately should make the process ergodic. However, a rigorous proof is missing as the addition of almost any type of noise makes the noisy model non-reversible. This makes it difficult to apply methods developed for the East model, as they heavily rely on the reversibility of the process and the fact that the invariant measure is explicitly known.

The proof, as described in the introduction, is a coupling argument that leverages the presence of the long-lived state $a \in \mathcal{A}$, which is unlikely to flip under any configuration of the neighbors. Theorem 3.1 assumes the symbol a spontaneously appears in all coupled trajectories with some frequency, for example, by the positive rates property. This agreement persists for a long time, blocking the spread of disagreements between trajectories. Hence, if the ratio between the rates of births and deaths of the symbol a is sufficiently high, then disagreements fail to percolate and die out. This leads to local coalescence between all coupled trajectories and in turn ergodicity.

The method does not fundamentally require the assumptions of one-sided, nearest-neighbor interactions. To drop one-sidedness assumption one would have to modify the random walks constructed in section 4 to trace the non-monotone boundaries of the region of disagreement. To drop assumption of nearest-neighbor interactions the long lived state must appear more frequently, to create blocks of agreement of width greater or equal to interaction range. The restriction to one-dimensional IPS seems to be a hard limit, as in higher dimensions, a disagreement can easily go around a blocked site.

4 Proof of main result

The proof is via a coupling argument. Recall that $\mathcal{A} = \{a_1, \dots, a_n\}$ is a finite set called an alphabet, the elements of which we call states. For any initial configuration $\zeta_0 \in \mathcal{A}^{\mathbb{Z}}$, the trajectory of the IPS is denoted by $\{\zeta_t\}_{t \geq 0}$. We consider the canonical coupling: if we let $(\tau_n^j)_{n \geq 0}^{j \in \mathbb{Z}}$ be the stopping times of the n -th update at site $j \in \mathbb{Z}$, and let $(U_n^j)_{n \geq 0}^{j \in \mathbb{Z}}$ be i.i.d. uniformly distributed random variables on $[0, 1]$. Then we set

$$\begin{aligned} \zeta_{\tau_n^j}(j) &:= a_k \text{ if } U_n^j \in \left[\sum_{m=1}^{k-1} P_j(a_m | \zeta_{\tau_n^{j-}}), \sum_{m=1}^k P_j(a_m | \zeta_{\tau_n^{j-}}) \right) \\ \zeta_t(j) &:= \zeta_{\tau_n^j}(j) \text{ for } t \in [\tau_n^j, \tau_{n+1}^j). \end{aligned} \quad (4.1)$$

At an update time $t = \tau_n^j$, the probability that the new state at site j equals a_k , given the left limit ζ_{t-} of the trajectory, is $P_j(a_k | \zeta_{t-})$. This construction requires the existence of left limits ζ_{t-} . We refer a reader wishing to verify that an IPS is a càdlàg process to Chapter 1, Section 3 of [9]. This construction defines the trajectories of all initial configuration on a shared probability space, i.e. the same choice of $(\tau_n^j)_{n \geq 0}^{j \in \mathbb{Z}}$ and $(U_n^j)_{n \geq 0}^{j \in \mathbb{Z}}$. In order to study the ergodicity of the IPS, we introduce the family of stopping times $(\pi_j)_{j \in \mathbb{Z}}$ given by

$$\pi_j := \sup_{\zeta_0 = \xi_0 \text{ off } j} \inf\{t \geq 0 : \zeta_t = \xi_t\}.$$

The stopping time π_j is the first time after which for all pairs of initial configurations that only differed at site j their trajectories converged. Hence, the system forgot the initial state at the site j . The following lemma (Theorem B of [10]) makes the connection between this notion and ergodicity:

Lemma 4.1. *If an IPS on $\mathbb{Z}^d$ has a finite range of interactions and $\lim_{t \rightarrow \infty} t^d \mathbb{P}(\pi_0 > t) = 0$, then the IPS is uniformly exponentially ergodic.*

To explain the statement Lemma 4.1 we need to recall the definition of *cone of dependence*. Intuitively, for a point in space-time $(j, t) \in \mathbb{Z}^d \times \mathbb{R}_+$ the $\mathbf{Cone}(j, t)$ is the set of all points in the past of t that could have communicated with the point (j, t) . For formal construction, we point to Section 2.1 in [6]. The meaning of Lemma (4.1) is this: the distribution of the IPS over a fixed finite set $S \subset \mathbb{Z}^d$ at time t depends on the initial configuration only through the sites at time zero, that are in the cone of dependence of set S . Their number grows like $O(t^d)$, so if the rate at which each is forgotten is faster then the distribution over sites in S becomes independent from the initial configuration. Ergodicity follows.

To prove Theorem 3.1, we use a random walk argument. The idea is to construct a pair of random walks $\{Y_n\}_{n \in \mathbb{N}}$ and $\{X_n\}_{n \in \mathbb{N}}$ that act as left and right boundaries for disagreements, i.e., the region in between them should contain all the points (j, t) such that there exists $\zeta_0 = \xi_0$ off 0 with $\zeta_t(j) \neq \xi_t(j)$. If the RWs cross, then disagreements die out. Assuming good control on how fast that occurs, Lemma 4.1 can be employed. The following lemma makes this idea precise.

Lemma 4.2. *Suppose for a one-dimensional, one-sided IPS there exist random walks $\{X_n\}_{n \in \mathbb{N}}$ and $\{Y_n\}_{n \in \mathbb{N}}$ on $\mathbb{R}_+$ with the following properties:*

1. *The RW Y_n is not slower than the spread of information from 0:*

$$Y_n \geq \inf\{t \in \mathbb{R}_+ : 0 \in \mathbf{Cone}(-n, t)\}.$$

The proof requires that $Y_n \rightarrow \infty$ to ensure that once the processes $\{X_n\}_{n \in \mathbb{N}}$ and $\{Y_n\}_{n \in \mathbb{N}}$ are high enough they cannot get down without crossing. The condition above realizes this for free.

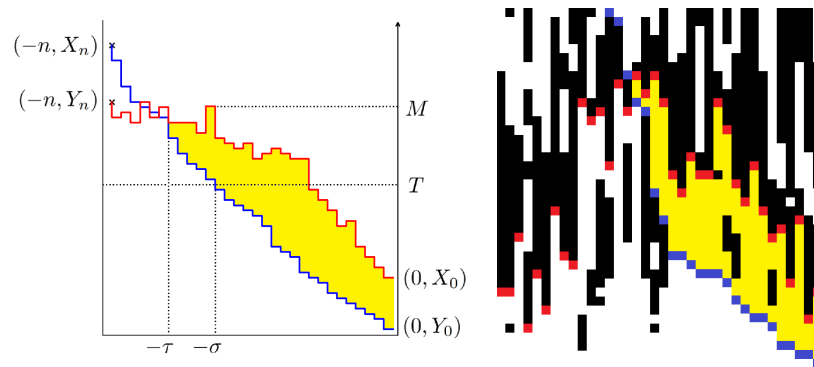


Figure 3: Left: representation of the random walks with the red and blue lines tracing points $(-n, X_n)$ and $(-n, Y_n)$ respectively. Initially, the RWs may diverge, but after they pass threshold T , they tend to cross. The yellow region marks the points in spacetime where disagreements may occur, with the initial disagreement only present at site zero. Right: Simulation of the random walks $\{X_n\}_{n \in \mathbb{N}}$ and Y_n (as constructed in the proof of Theorem 3.1) with $\beta = \delta = 0.1$. The arrow of time points up, the X -axis represents sites in range $\{-40, \dots, 0\}$. The points in spacetime that lie in the intervals of agreement (set $\bigcup_{j \in \mathbb{Z}} \{j\} \times A_j$) are colored black. The red marks the points $(-n, X_n)$ and the blue points $(-n, Y_n)$. The region where disagreements may occur is marked in yellow, with the initial disagreement only present at site zero.

2. The random walks bound the region with disagreements (cf. Figure 3):

$$\bigcup_{\zeta_0 = \xi_0 \text{ off } 0} \{(j, t) \in \mathbb{Z} \times \mathbb{R}_+ : \zeta_t(j) \neq \xi_t(j)\} \subseteq \bigcup_{n \geq 0} \{-n\} \times (Y_n, X_n].$$

3. The walk $Z_n := X_n - Y_n$ is eventually a supermartingale with negative drift, i.e. for some $\varepsilon > 0$ there exists a $T > 0$ such that

$$\mathbb{E}[Z_{n+1} - Z_n \mid \mathcal{F}_n] \leq -\varepsilon$$

on the event $\{X_n, Y_n \geq T\}$. Here $\mathcal{F}_n$ is the canonical filtration for the process (X_n, Y_n) .

4. The increments of X_n and Y_n are controlled: There are constants $A, \alpha \geq 1$ such that

$$\begin{aligned} \mathbb{E}[|X_{n+1} - X_n|^k \mid \mathcal{F}_n] &\leq Ak! \alpha^k, \\ \mathbb{E}[|Y_{n+1} - Y_n|^k \mid \mathcal{F}_n] &\leq Ak! \alpha^k, \text{ for all } k \geq 0. \end{aligned}$$

5. There exists $\lambda > 0$ such that $\mathbb{E}[e^{\lambda X_0}] < \infty$.

Then the IPS is uniformly, exponentially ergodic.

Proof of Lemma 4.2. We aim to apply Lemma 4.1. To find suitable bounds on the tail of the distribution of π_0 , we define the random variables

$$\begin{aligned} \tau &:= \inf \{n \in \mathbb{N} : X_n \leq Y_n\} = \inf \{n \in \mathbb{N} : Z_n \leq 0\}, \\ M &:= \sup_{n \leq \tau} X_n. \end{aligned}$$

The variable M is the maximal time that the region between $\{Y_n\}_{n \in \mathbb{N}}$ and $\{X_n\}_{n \in \mathbb{N}}$ reaches, so by assumption (2) we have $\pi_0 \leq M$. To verify the condition $t\mathbb{P}(\pi_0 > t) \rightarrow 0$, it thus suffices to check that $\mathbb{E}[M] < \infty$. The following estimate shows that all we need is a good control over the tail of τ :

$$\begin{aligned} \mathbb{E}[M] &\leq \mathbb{E}[X_0] + \mathbb{E}\left[\sum_{n=0}^{\tau} |X_{n+1} - X_n|\right] \\ &= \mathbb{E}[X_0] + \sum_{n=0}^{\infty} \mathbb{E}[|X_{n+1} - X_n| \mathbf{1}(\tau \geq n)] \\ &\leq \mathbb{E}[X_0] + \sum_{n=0}^{\infty} \mathbb{E}\left[|X_{n+1} - X_n|^2\right]^{\frac{1}{2}} \mathbb{P}(\tau \geq n)^{\frac{1}{2}} \\ &\leq \mathbb{E}[X_0] + (2A)^{\frac{1}{2}} \alpha \sum_{n=0}^{\infty} \mathbb{P}(\tau \geq n)^{\frac{1}{2}}. \end{aligned} \quad (4.2)$$

Here we used Hölder's inequality and the assumption (4). By the assumption (5) the $\mathbb{E}[e^{\lambda X_0}]$ is finite hence so is $\mathbb{E}[X_0]$. To conclude that $\mathbb{E}[M] < \infty$ we show that

$$\mathbb{P}(\tau > n) = O(e^{-\gamma n}),$$

for some $\gamma > 0$. To estimate the tail of τ precisely, we introduce one more stopping time:

$$\sigma := \inf \{n \in \mathbb{N} : 0 \notin \mathbf{Cone}(-n, T)\}.$$

The variable $-\sigma$ is the right-most negative site that had no chance to communicate with the site 0 up to time T (cf. assumption 3). By assumption (1) it is guaranteed that $Y_n \geq T$ for $n \geq \sigma$. Thus the stopping time σ is the time after which the size of disagreement interval starts acting like a supermartingale ($X_\sigma \geq Y_\sigma \geq T$ + assumption (3)), or is already zero ($X_\sigma < Y_\sigma$). Because we only know that the walk Z_n becomes a supermartingale *eventually* we define its shifted modification $S_n := Z_{\sigma+n}$. To simplify notation we also introduce $\rho := \max\{0, \tau - \sigma\}$, i.e. the stopping time of S_n passing below zero and $\mathcal{G}_n$, the σ -algebras generated by the RW S_n and $\mathcal{F}_\sigma$. By assumption (3) we have $\mathbb{E}[S_{n+1} - S_n | \mathcal{G}_n] \leq -\varepsilon$, on the event $\{S_n \geq 0\}$, i.e. the RW S_n acts like a supermartingale with negative drift until it passes below zero. To conclude that the distribution of ρ has an exponentially decaying tail we aim to use Theorem 2.3 (inequality 2.9) from [8]. The condition $C1$ of [8] is the same as our assumption (3). Condition $D2$, implied by $\sup \mathbb{E}[e^{\lambda|S_{n+1}-S_n|} | \mathcal{G}_n] < \infty$ is met for small λ due to assumption (5). Lastly we verify the condition $D1$, i.e. that for some $\gamma > 0$ and $\lambda > 0$ we have $\mathbb{E}[e^{\lambda(S_{n+1}-S_n)} | \mathcal{G}_n] - e^{-\gamma} \leq 0$, on the event $\{S_n \geq 0\}$ (the quantity $e^{-\gamma}$ is the equivalent of ρ in the notation of [8]). Using assumptions (3) and (4) we observe that for λ sufficiently small

$$\begin{aligned} \mathbb{E}[e^{\lambda(S_{n+1}-S_n)} | \mathcal{G}_n] &= 1 + \lambda \mathbb{E}[S_{n+1} - S_n | \mathcal{G}_n] + \sum_{k \geq 2} \frac{\lambda^k}{k!} \mathbb{E}[(S_{n+1} - S_n)^k | \mathcal{G}_n] \\ &\leq 1 - \lambda\varepsilon + A(e^{\lambda\alpha} - \lambda\alpha - 1) \\ &= 1 - \lambda\varepsilon + o(\lambda^2) \end{aligned}$$

If we also fix γ small enough so that $\mathbb{E}[e^{\lambda(S_{n+1}-S_n)} | \mathcal{G}_n] \leq e^{-\gamma}$, then conditions of Theorem 2.3 of [8] are met and $\mathbb{P}(\rho > n | \mathcal{G}_0) \leq e^{-n\gamma} e^{\lambda S_0}$ holds for some $\lambda, \gamma > 0$. In

terms of the random walk Z_n and stopping times σ and τ , this implies

$$\begin{aligned}\mathbb{P}(\sigma + n < \tau) &= \mathbb{E}[\mathbb{P}(\rho > n \mid \mathcal{G}_0)] \\ &\leq e^{-n\gamma} \mathbb{E}[e^{\lambda Z_\sigma}] \\ &\leq e^{-n\gamma} \mathbb{E}[e^{\lambda X_\sigma}].\end{aligned}$$

If the $\mathbb{E}[e^{\lambda X_\sigma}]$ is finite for some $\lambda > 0$, then $\mathbb{P}(\sigma + n < \tau)$ decays exponentially. To show that this expectation is finite, we must understand the distribution of σ . The position of the left-most site that has 0 in its cone of dependence moves with a rate-one Poisson point process. Hence, if we let $\{N_t\}_{t \geq 0}$ be the Poisson point process σ has the distribution of N_T . Returning to $\mathbb{E}[e^{\lambda X_\sigma}]$ we estimate via Hölder

$$\begin{aligned}\mathbb{E}[e^{\lambda X_\sigma}] &\leq \sum_{n=0}^{\infty} \mathbb{P}(\sigma = n)^{\frac{1}{2}} \mathbb{E}[e^{2\lambda X_n}]^{\frac{1}{2}} \\ &= \sum_{n=0}^{\infty} \left(\frac{T^n}{n!}\right)^{\frac{1}{2}} \mathbb{E}[e^{2\lambda X_n}]^{\frac{1}{2}}\end{aligned}\tag{4.3}$$

To show that this is finite, we find an upper bound for $\mathbb{E}[e^{2\lambda X_n}]$:

$$\begin{aligned}\mathbb{E}[e^{2\lambda X_n}] &= \mathbb{E}\left[e^{2\lambda X_{n-1}} \mathbb{E}[e^{2\lambda(X_n - X_{n-1})} \mid \mathcal{F}_{n-1}]\right] \\ &\leq \mathbb{E}\left[e^{2\lambda X_{n-1}} \sum_{k=0}^{\infty} \frac{(2\lambda)^k}{k!} \mathbb{E}[|X_n - X_{n-1}|^k \mid \mathcal{F}_{n-1}]\right] \\ &\leq \mathbb{E}[e^{2\lambda X_{n-1}}] \frac{A}{1 - 2\alpha\lambda} \\ &\leq \dots \\ &\leq \mathbb{E}[e^{2\lambda X_0}] \left(\frac{A}{1 - 2\alpha\lambda}\right)^n.\end{aligned}$$

In the third step, we used the assumption (4), and the fourth is induction on n . Setting the $\lambda < \frac{1}{2\alpha}$ and sufficiently small so that $\mathbb{E}[e^{2\lambda X_0}] < \infty$ (cf. assumption (5)) yields that growth of $\mathbb{E}[e^{2\lambda X_n}]$ is only exponential in n . Thus the $\frac{1}{\sqrt{n!}}$ term in (4.3) guarantees that the sum converges and $C := \mathbb{E}[e^{\lambda X_\sigma}]$ is finite. We have obtained

$$\mathbb{P}(\sigma + n < \tau) \leq C e^{-n\gamma}$$

for some positive γ . This, in combination with a Chernoff bound on the tail of Poisson distributed σ allows us to conclude that $\mathbb{P}(n < \tau)$ decays exponentially:

$$\begin{aligned}\mathbb{P}(n < \tau) &\leq \mathbb{P}\left(\sigma < \frac{1}{2}n, n < \tau\right) + \mathbb{P}\left(\sigma \geq \frac{1}{2}n\right) \\ &\leq \mathbb{P}\left(\sigma + \frac{1}{2}n < \tau\right) + \mathbb{P}\left(\sigma \geq \frac{n}{2}\right) \\ &\leq C e^{-\frac{1}{2}n\gamma} + e^{-T} \left(\frac{2eT}{n}\right)^{\frac{n}{2}} \\ &= O\left(e^{-\frac{1}{2}\gamma n}\right).\end{aligned}$$

By the estimate in (4.2) exponential decay of the tail of τ implies M has a finite expectation and thus $t\mathbb{P}(M > t) \rightarrow 0$. The inequality $\pi_0 \leq M$ finishes the proof, by Lemma (4.1). $\square$

The proof of Theorem 3.1 is a straightforward application of Lemma 4.2 to random walks tailored to the considered IPS. Assume without loss of generality that $\mathcal{A} = \{0, \dots, |\mathcal{A}| - 1\}$ and the long-lived state a is 0. Let ζ_0 and ξ_0 be any two initial configurations, whose trajectory we couple in the way described in (4.1). The following observation is crucial for our result:

(a) If $U_n^j < \beta = \min_{\zeta} P_j(0 \mid \zeta)$, then

$$\zeta_t(j) = \xi_t(j) = 0 \quad \text{for } t \in [\tau_n^j, \tau_{n+1}^j).$$

(b) If $\zeta_{\tau_n^j-}(j) = \xi_{\tau_n^j-}(j) = 0$ and $U_n^j < 1 - \delta = \min_{\zeta: \zeta(j)=0} P_j(0 \mid \zeta)$ then

$$\zeta_t(j) = \xi_t(j) = 0 \quad \text{for } t \in [\tau_n^j, \tau_{n+1}^j).$$

The above properties mean that at every site, one in $\frac{1}{\beta}$ updates causes an agreement of zeroes to appear, shared among all trajectories, and this agreement survives for an average of at least $\frac{1}{\delta}$ updates. This happens independently at every site. We observe that disagreement can't cross these blocks due to nearest-neighbor interaction. Hence, if those agreements of zero cover a sufficiently high proportion of spacetime, then disagreement does not percolate, and hence the IPS is ergodic. The following proof makes this heuristic argument rigorous.

Proof of Theorem 3.1. We aim to apply Lemma 4.2. We first mark a subset of spacetime where the coupled processes must agree and then construct two random walks $\{X_n\}_{n \in \mathbb{N}}$ and $\{Y_n\}_{n \in \mathbb{N}}$ that bound the disagreement. For an illustration, we refer to Figure 3. We first set A_j to be a subset of $\mathbb{R}_+$ made of intervals $[\tau_n^j, \tau_m^j]$, $m > n$ such that

$$U_n^j < \beta, \quad U_k^j < 1 - \delta \text{ for } n \leq k < m.$$

The first condition guarantees that an agreement of zeroes appears at (j, τ_n^j) and the second that the agreement persists for the next $m - n$ updates at j . Then, we have an agreement of zeroes on every spacetime block $\{j\} \times A_j$. With that, we set the first random walk

$$X_0 := \inf A_0, \\ X_{n+1} := \begin{cases} \sup A_{-(n+1)}^c \cap [0, X_n], & X_n \in A_{-(n+1)} \\ \inf A_{-(n+1)} \cap [X_n, \infty], & X_n \notin A_{-(n+1)}. \end{cases}$$

The random walk $\{X_n\}_{n \in \mathbb{N}}$ at every step checks if the point $(-(n+1), X_n)$ is one of the points where all trajectories agree to have a zero state. If so, then X_{n+1} traces down the beginning of the interval of agreement in which $(-(n+1), X_n)$ lies. Else X_n goes up until a new interval of agreement appears. For the other random walk, we set

$$Y_0 := 0, \\ Y_{n+1} := \begin{cases} \inf A_{-(n+1)}^c \cap [Y_n, \infty], & Y_n \in A_{-(n+1)} \\ \min\{\tau_k^{-(n+1)} \geq Y_n\}, & Y_n \notin A_{-(n+1)}. \end{cases}$$

The random walk $\{Y_n\}_{n \in \mathbb{N}}$ similarly checks whether the point $(-(n+1), Y_n)$ is in an interval of agreement, and if it is, then Y_{n+1} increments up to when this interval ends. Else Y_{n+1} increments to the next time there is any update at $-(n+1)$, as this is the earliest time a disagreement might spread to that site.

If trajectories $\{\zeta_t\}_{t \geq 0}$ and $\{\xi_t\}_{t \geq 0}$ only differ at site zero at time zero, then the set of points in spacetime where they disagree is contained in the set

$$\bigcup_{n \in \mathbb{N}} \{-n\} \times (Y_n, X_n].$$

The RWs satisfy the Assumptions (1), (2) and (5) of Lemma 4.2. The only remaining ingredient is the estimates on the increments of $\{X_n\}_{n \in \mathbb{N}}$ and $\{Y_n\}_{n \in \mathbb{N}}$. The verification of Assumption (4) of the lemma is straightforward. The increments of $\{X_n\}_{n \in \mathbb{N}}$ and $\{Y_n\}_{n \in \mathbb{N}}$ are dominated by $\text{Exp}(\min\{\beta, \delta\})$ random variables, so we have

$$\begin{aligned} \mathbb{E} \left[|X_{n+1} - X_n|^k \right] &\leq \frac{k!}{\min\{\beta, \delta\}^k}, \\ \mathbb{E} \left[|Y_{n+1} - Y_n|^k \right] &\leq \frac{k!}{\min\{\beta, \delta\}^k}. \end{aligned}$$

Now we verify the Assumption (3). We compute the expected increments of $\{Y_n\}_{n \in \mathbb{N}}$ first. To do that, we need to find $\mathbb{P}(t \in A_j)$. We first compute its derivative

$$\begin{aligned} \frac{\partial}{\partial t} \mathbb{P}(t \in A_j) &= -\delta \mathbb{P}(t \in A_j) + \beta \mathbb{P}(t \notin A_j) \\ &= -(\beta + \delta) \mathbb{P}(t \in A_j) + \beta. \end{aligned}$$

Since $\mathbb{P}(0 \in A_j) = 0$ then

$$\mathbb{P}(t \in A_j) = \frac{\beta}{\beta + \delta} - \frac{\beta}{\beta + \delta} e^{-(\beta + \delta)t}.$$

By the memoryless property of the exponential distribution, we have

$$\begin{aligned} \mathbb{E}[Y_{n+1} - Y_n \mid Y_n] &= \frac{1}{\delta} \mathbb{P}(t \in A_{-(n+1)} \mid t=Y_n) + \mathbb{P}(t \notin A_{-(n+1)} \mid t=Y_n) \\ &= \frac{\beta + \delta^2}{\delta(\beta + \delta)} + O\left(e^{-(\beta + \delta)Y_n}\right). \end{aligned} \quad (4.4)$$

To calculate $\mathbb{E}[X_{n+1} - X_n \mid X_n]$, we consider two cases: either $X_n \notin A_{-(n+1)}$ or $X_n \in A_{-(n+1)}$. In the first case $X_{n+1} - X_n$ has the $\text{Exp}(\beta)$ distribution and we have

$$\mathbb{E}[(X_{n+1} - X_n) \mathbb{1}(X_n \notin A_{-(n+1)}) \mid X_n] = \frac{\delta}{\beta(\beta + \delta)} + O\left(e^{-(\beta + \delta)X_n}\right). \quad (4.5)$$

To compute the contribution to the drift of $\{X_n\}_{n \in \mathbb{N}}$ from the jumps down, we introduce the random variable that for each point in spacetime (j, t) marks the last time before t that a zero-killing update occurred at site j , if there is any:

$$\tau_\delta(j, t) := \sup \left\{ \tau_n^j \leq t : U_n^j \geq 1 - \delta \right\} \cup \{0\}.$$

If a zero-creating ($U_n^j < \beta$) update occurs at j in the interval $[\tau_\delta(j, t), t]$ then $t \in A_j$. Let $\tau_\beta(j, t)$ be the time of first zero-creating update after $\tau_\delta(j, t)$, i.e.

$$\tau_\beta(j, t) := \inf \left\{ \tau_n^j > \tau_\delta(j, t) : U_n^j < \beta \right\}.$$

Then we can express the contribution of jumps down

$$\mathbb{E}[(X_{n+1} - X_n) \mathbb{1}(X_n \in A_{-(n+1)}) \mid X_n]$$

using the new random variables as

$$\mathbb{E} [\min \{0, \tau_\beta(-(n+1), X_n) - X_n\} \mid X_n].$$

To calculate this expectation, we only need to know the distribution of

$$\tau_\beta(j, t) - t,$$

for arbitrary t . The distribution does not depend on j . We can represent this random variable as a sum of two random variables with known distributions, which we argue are independent:

$$\tau_\beta(j, t) - t = [\tau_\beta(j, t) - \tau_\delta(j, t)] + [\tau_\delta(j, t) - t].$$

The zero-killing updates arrive with rate δ , so the distribution of $t - \tau_\delta(j, t)$ is that of $\min\{t, W\}$, where W is an $\text{Exp}(\delta)$ random variable. Because the zero-killing and zero-creating updates arrive independently (independence follows by Poisson thinning) the increment $\tau_\beta(j, t) - \tau_\delta(j, t)$ has distribution $\text{Exp}(\beta)$ and is independent of $\tau_\delta(j, t)$. The contribution to drift of X_n from jumps down is thus

$$\int_{\mathbb{R}_+^2} \beta \delta e^{-\beta x - \delta y} \min\{0, x - \min\{X_n, y\}\} d(x, y) = -\frac{\beta}{\delta(\beta + \delta)} + O(e^{-\delta X_n}). \quad (4.6)$$

By the combination of (4.4), (4.5) and (4.6) the RW $Z_n := X_n - Y_n$ satisfies condition (3) of Lemma 4.2 if

$$-\frac{\beta + \delta^2}{\delta(\beta + \delta)} + \frac{\delta}{\beta(\beta + \delta)} - \frac{\beta}{\delta(\beta + \delta)} < 0. \quad (4.7)$$

To finish off the proof, we apply the Time-Scaling Lemma. When the IPS is slowed down, the coefficients β and δ converge to zero, but the ratio $\frac{\beta}{\delta}$ remains invariant. Thus (4.7) simplifies to

$$-2\frac{\beta}{\delta} + \frac{\delta}{\beta} < 0,$$

which is equivalent to $\delta < \sqrt{2}\beta$. □

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